An ILB- Manifold Structure on the Set of Riemannian Metrics on a Noncompact Manifold

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Abstract

In this paper, using the structures of cone and bicone fields on vector bundles, the author introduces a ILB (inverse limit of Banach)-manifold structure on $\mathcal{M}$ the space of Riemannian metrics on a non-compact manifold M. In the last section, it is proven that, this way, on the open submanifold $\mathcal{M}_{finite}$ of finite volume metrics, the canonical Riemannian metric is defined.

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1 Preliminaries

First, let M be a topological manifold, paracompact, with $\partial M = \emptyset$, which need not to be compact. Let (E, p, M) be a topological vector bundle over M.

Definition 1.1. [PA-cones]
A *cone field* on the vector bundle (E, p, M) is a map $K : M \to \mathcal{P}(E)$, $x \mapsto K(x) \subset E_x$ which satisfies the following two conditions:
(K1) $(\forall) x \in M,\ K(x)$ is a convex cone, closed in E_x, pointed, solid;
(K2) $\cup_{x\in M} Int(K(x))$ and $\cup_{x\in M}(E_x \backslash K(x))$ are open in E.

In the definition above, a *convex cone* is, following [KRA], a set K which satisfies $K + K \subset K$ and $(\forall)\lambda \geq 0,\ \lambda K \subset K$. A cone K which satisfies $K \cap -K = \{0\}$ will be called *pointed cone*, and *a solid cone* is a cone which has interior points in the topology of E_x.

The structure consisting by a vector bundle (E, p, M) and a cone field K on it is denoted by $[(E, p, M); K]$.

Example 1.2. [PA-metrics] Let us consider now the bundle $(\mathcal{S}^2T^*M, p, M)$ of 2 times covariant symmetric tensors on a given manifold M. We put $(\forall)x \in M$

$$K_k(x) := \{t_x \in \mathcal{S}^2T^*M_x | r(t_x) = i_p(t_x)\},$$

where r denotes the rank and i_p denotes the positive inertia index. Then $x \mapsto K(x) := \cup_{k=1}^{n} K_k(x)$ defines a cone field on the bundle $(\mathcal{S}^2T^*M, p, M)$.

There are local and global properties of this structure, exposed in [PA-cones].

Consider now $\Gamma^0(E)$, the space of continuous sections of the bundle (E, p, M).

Definition 1.3. [PA-cones]
We call a *positive section* of the structure $[(E, p, M); K]$, a section $\sigma \in \Gamma^0(E)$ for which $\sigma(x) \in K(x),\ (\forall)x \in M$.

The set of positive sections is denoted by K_Γ^0; if on the space $\Gamma^0(E)$ is considered the graph topology WO^0 then we have:

Proposition 1.4. *[PA-cones]*
The set K_Γ^0 is a convex cone, pointed, solid, in $\Gamma^0(E)$. Moreover, K_Γ^0 is a generating cone in $\Gamma^0(E)$, i.e. $\forall \sigma \in \Gamma^0(E)$ $(\exists)\zeta_1, \zeta_2 \in K_\Gamma^0$ such that $\sigma = \zeta_1 - \zeta_2$.

The cone K_Γ^0 defines a partial order relation on $\Gamma^0(E)$ by

$$\sigma_1, \sigma_2 \in \Gamma^0(E),\ \sigma_1 \leq \sigma_2 \overset{def}{\Longleftrightarrow} \sigma_2 - \sigma_1 \in K_\Gamma^0. \tag{1}$$

Proposition 1.5. `Papuc`*[PA-cones]*
The pair $(\Gamma^0(E), \leq)$ is an ordered vector space, directed on both sides. Endowed with the WO^0 topology is a topological vector space iff M is a compact manifold.

Given a fixed $\zeta \in Int(K_\Gamma^0)$, we denote by Γ_ζ^0 the set of $\zeta-$ measurable elements of $\Gamma^0(E)$:

$$\Gamma_\zeta^0 := \{\sigma \in \Gamma^0(E) \mid (\exists)\lambda \in \mathbb{R}_+ : -\lambda\zeta \leq \sigma \leq \lambda\zeta\}, \tag{2}$$

and we consider the map

$$|\cdot|_\zeta^0 : \Gamma_\zeta^0 \to \mathbb{R},\ |\sigma|_\zeta := min_{\lambda \in \mathbb{R}_+}\{-\lambda\zeta \leq \sigma \leq \lambda\zeta\}. \tag{3}$$

Proposition 1.6. *[PA-cones]*
1. $\Gamma^0_\zeta = \Gamma^0(E)$ *iff* M *is a compact manifold;*
2. The map $|\cdot|^0_\zeta$ *defined by equation (3) from above is a norm on* Γ^0_ζ*;*
3. The set of all Γ^0_ζ *is a covering of* $\Gamma^0(E)$*;*
4. If $\Gamma^0_c(E)$ *denotes the subspace of compact support sections, then* $\Gamma^0_c(E) \subset \Gamma^0_\zeta, (\forall)\zeta \in Int(K^0_\Gamma)$*;*
5. If $\zeta \in \Gamma^0_{\zeta_1}$*, with* $\zeta, \zeta_1 \in Int(K^0_\Gamma)$ *then* $\Gamma^0_\zeta \subset \Gamma^0_{\zeta_1}$.

Theorem 1.7. *[VA] If* $\zeta \in K^0_\Gamma$*, then* $(\Gamma^0_\zeta, |\cdot|^0_\zeta)$ *is a Banach space.*

We consider now (E, p, M), a $\mathcal{C}^k$- differentiable bundle over M, a $\mathcal{C}^k$-differentiable manifold, $k \geq 1$, which need not to be compact.

Definition 1.8. [VA-bicone] A *bicone field* on a vector bundle (E, p, M) is the structure consisting of a cone field K on the bundle (E, p, M) and a second cone field K_{TM} on the tangent bundle (TM, p, M).

We will denote by $[(E, p, M); K; K_{TM}]$ the structure consisting of a bicone field on the vector bundle (E, p, M).

The existence of a bicone field on a vector bundle (E, p, M) is equivalent with the existence of a non zero section $\zeta \in \Gamma^0(E)$ and of a nonzero vector field on M.

Now, as a consequence of the vector bundle isomorphism

$$J^k E \cong \oplus_{i=1}^k \mathcal{L}_s(TM^i, E) \tag{4}$$

from [Pal] page 90, we have

Proposition 1.9. *[VA-bicone] If* $[(E, p, M); K; K_{TM}]$ *is a* $\mathcal{C}^p$*- differentiable vector bundle endowed with a bicone field then the vector bundle* $(J^k E, p, M)$ *is endowed in a natural way with a cone field* K^k*,* $(\forall)k \leq p$.

Next, as usually, we will denote by convention $J^0 E := E,\ j^0\zeta := \zeta$.

Definition 1.10. [VA-bicone] A section $\zeta \in \Gamma^k(E)$ which satisfies $\zeta(x) \in K(x)$ and $j^i\zeta(x) \in K^i(x), i = \overline{0,k}, (\forall)x \in M$ will be called section *positive up to order* k.

The set of positive sections up to order k will be denoted by K^k_Γ. On $\Gamma^k(E)$, the space of $\mathcal{C}^k$- differentiable sections we will consider the Whitney WO^k- topology, which on a space of sections can be given by a base of neighborhoods $W(\sigma_0, U)$, where $\sigma_0 \in \Gamma^k(E)$ and U is an open neighborhood of $Im(j^k\sigma_0)$ in $J^k(E)$:

$$W(\sigma_0, U) := \{\sigma \in \Gamma^k(E) \mid j^k\sigma(x) \in U, (\forall)x \in M\}. \tag{5}$$

Proposition 1.11. *K_Γ^k is a convex cone, closed, pointed and solid in the space $(\Gamma^k(E), WO^k)$.*

Corollary 1.12. *[KRA] The cone K_Γ^k defines on $\Gamma^k(E)$ an order relation by $\sigma_1 \leq \sigma_2 \stackrel{def}{\Longleftrightarrow} \sigma_2 - \sigma_1 \in K_\Gamma^k$. In particular, this relation is directed on both sides.*

Let $\zeta \in Int(K_\Gamma^k)$ be fixed.

Definition 1.13. [VA-bicone] A section $\sigma \in \Gamma^k(E)$ for which exists $\lambda \in \mathbb{R}_+$ s.t.

$$-\lambda j^i\zeta \leq j^i\sigma \leq \lambda j^i\zeta,\ i = \overline{1,k}$$

will be called ζ- *measurable up to order k.*

As in [PA-cones], we have that the map

$$|\cdot|_\zeta^k : \Gamma_\zeta^k \to \mathbb{R}_+,\ |\sigma|_\zeta^k := \min\{\lambda \in \mathbb{R}_+ \mid -\lambda j^i\zeta \leq j^i\sigma \leq \lambda j^i\zeta,\ i = \overline{1,k}\}$$

is a norm on the vector space Γ_ζ^k of ζ- measurable sections up to order k, and with this norm, Γ_ζ^k becomes a Banach space (the proof is absolutely similar to the one from [VA]). The open ball in the norm $|\cdot|_\zeta^k$, centered in σ, of radius ϵ, will be denoted by $B_\zeta^k(\sigma, \epsilon)$ and as in [PA-cones], coincides with the open centered intervals in the order relation from corollary 1.12.

Let us denote now by τ^k the topology on $\Gamma^k(E)$ obtained by taking the path connected components of the WO^k- topology.

Theorem 1.14. *[VA-bicone] For all $k \in \mathbb{N}$, the τ^k- topology on $\Gamma^k(E)$ is the topology for which a basis of neighborhoods is given by*

$$\{B_\zeta^k(\sigma, \epsilon) \mid \zeta \in Int(K_\Gamma^k),\ \sigma \in \Gamma_\zeta^k,\ \epsilon \gneq 0\}.$$

2 The ILB- manifold Structure on the Space of Riemannian Metrics

Let now (E, p, M) be a smooth vector bundle, endowed with a bicone field defined by the cone fields K, K_{TM}.

Definition 2.1. [VA-bicone] A smooth section $\zeta \in \Gamma(E)$ which satisfies $\zeta(x) \in K_\Gamma^k,\ (\forall) k \in \mathbb{N}$ will be called a *indefinitely positive section.*

We will denote by K_Γ the set of indefinitely positive sections.

Proposition 2.2. *[VA-bicone] The set K_Γ is a (nonempty) pointed closed convex cone in* $(\Gamma(E), WO^\infty)$.

Corollary 2.3. *[VA-bicone] On* $\Gamma(E)$ *there is an order relation defined by* $\sigma \leq \sigma' \overset{def}{\Longleftrightarrow} \sigma' - \sigma \in K_\Gamma(E)$.

Let $\zeta \in \cap_k Int_{WO^k} K_\Gamma^k$ (this set is nonempty, see [VA-bicone]).

Definition 2.4. [VA-bicone] A section $\sigma \in \Gamma(E)$ which is ζ- measurable $(\forall)k \in \mathbb{N}$ will be called an *indefinitely* ζ*- measurable section.*

The set $\Gamma_\zeta(E)$ of indefinitely ζ- measurable sections is nonempty (e.g. $\zeta \in \Gamma_\zeta$) and is a vector space.

Proposition 2.5. *[VA-bicone] The space* $\Gamma_\zeta(E)$ *is the projective limit of the Banach spaces* $\Gamma_\zeta^k(E)$.

Corollary 2.6. *[VA-bicone] The following assumptions hold:*
(i) $\Gamma_\zeta(E)$ *is a complete, locally convex space;*
(ii) The τ^∞*- topology on* $\Gamma(E)$ *is the topology for which a base of neighborhoods is given by the set*

$$\{B_\zeta^k(\sigma, \epsilon) |\ \zeta \in \cap_k Int_{WO^k} K_\Gamma^k,\ k \in \mathbb{N},\ \epsilon \gneq 0\};$$

(iii) The set $\{\Gamma_\zeta(E), \Gamma_\zeta^k(E) \mid\ k \in N(0)\}$ *is a ILB (inverse limit of Banach)-chain, following Omori's definition [OMORI], page 5.*

Since in the infinite dimensional geometry the notion of manifold might vary, we will refer in this paper to the notion from [MI-KRI], page170, for which the differences from the finite dimensional correspondent is that for each chart is allow a different model space, and the chart changing is require to be only smooth instead of smooth diffeomorphism.

Theorem 2.7. $\Gamma(E)$ *is a smooth manifold modelled by the ILB-spaces* $\Gamma_\zeta(E)$.

Proof. From [VA] and [VA-bicone] we have $\Gamma(E) = \varinjlim_\zeta \varprojlim_k \Gamma_\zeta^k$. The topology induced above on $\Gamma(E)$ is the τ^∞- topology. Then, again by the equation above, $\Gamma(E) = \cup_{\zeta \in Int(K_\Gamma)} \Gamma_\zeta(E)$.

Let $\sigma_0 \in \Gamma(E)$. There exists a positive section $\zeta_0 \in Int(K_\Gamma)$ such that $\sigma_0 \in \Gamma_{\zeta_0}(E) = \cap_k \Gamma_{\zeta_0}^k(E)$. Obviously, $U_{\zeta_0}(\sigma_0) := \cap_k B_{\zeta_0}^k(\sigma_0)$ is a nonempty open in τ^∞- topology neighborhood of σ_0. Let $\phi_{\sigma_0} : U_{\zeta_0}(\sigma_0) \subset \Gamma(E) \to \Gamma_{\zeta_0}$ be the restriction of the identity map $Id_{\Gamma_{\zeta_0}(E)}$. The pair $(U_{\zeta_0}(\sigma_0), \phi_{\zeta_0})$ is a chart around σ_0.

The charts changing is smooth. Indeed, Let $(U_{\zeta_1}(\sigma_1), \phi_{\sigma_1}), (U_{\zeta_2}(\sigma_2), \phi_{\sigma_2})$ be two charts with $U_{\zeta_1}(\sigma_1) \cup U_{\zeta_2}(\sigma_2) \neq \emptyset$. In particular, it follows that $U_{\zeta_1}(\sigma_1) \cup U_{\zeta_2}(\sigma_2) \subset \Gamma_{\zeta_1} \cap \Gamma_{\zeta_2}$. But from [PA-cones], the set $\{\Gamma_\zeta(E) | \zeta \in Int(K_\Gamma)\}$ is ordered and directed on both sides, by the inclusion. So there exists $\zeta_0 \in Int(K_\Gamma)$ such that $U_{\zeta_1}(\sigma_1) \cap U_{\zeta_2}(\sigma_2) \subset \Gamma_{\zeta_1} \cap \Gamma_{\zeta_2} \subset \Gamma_{\zeta_0}(E)$, and so the chart changing $\phi_{\sigma_2} \circ \phi_{\sigma_1}^{-1}$ is the restriction to an open set of the identity map $Id_{\Gamma_{\zeta_0}(E)}$, and so is smooth. Q.E.D.

Remark 2.8. In virtue of the example 1.2, $\Gamma(\mathcal{S}^2 T^* M)$, the space of two times covariant, symmetric tensor fields on the manifold M has the structure of a ILB-manifold, modelled by the spaces $\Gamma_g(\mathcal{S}^2 T^* M)$, with $g \in Int(K_\Gamma)$.

From [GiM-MICH] $\mathcal{M} = Int(K_\Gamma) \cap \Gamma(\mathcal{S}^2 T^* M)$, the space of all Riemannian metrics on the manifold M is τ^∞- open in $\Gamma(\mathcal{S}^2 T^* M)$.

Corollary 2.9. *The space $\mathcal{M}$ of all Riemannian metrics on M is an open submanifold of $\Gamma(\mathcal{S}^2 T^* M)$.*

3 The Riemannian Geometry of the Space of Riemannian Metrics of Finite Volume

We denote by $\mathcal{M}_{finite}$ the set of all Riemannian metrics of finite volume on M.

Remark 3.1. $\mathcal{M}_{finite}$ is τ^∞- open in $\mathcal{M}$. Indeed, let $(g_n)_{n \geq 0}$ a sequence of Riemannian metrics that converges in the τ^∞- topology to g_0, a finite volume metric. In particular, it follows that $(\forall) n \geq 0, g_n$ and g_0 differ only on a compact set, so each g_n is a finite volume metric.

On $\mathcal{M}$ there is a canonical Riemannian metric G, invariant under the natural action by pull- back of the group $Diff(M)$ of diffeomorphisms of M on $\mathcal{M}$, described in [EBIN], or [GiM-MICH]:

$$G_g : T_g\mathcal{M} \times T_g\mathcal{M} \to \mathbb{R},\ G_g(h,k) = \int_M trace(g^{-1}hg^{-1}k) d\nu_g, \tag{6}$$

To make clear the notation $g^{-1}hg^{-1}k$ we can regarde the bundle $\mathcal{S}^2(T^*M)$ as $\{h \in \mathcal{L}(TM, T^*M) |\ h^t = h\}$, subbundle of $\mathcal{L}(TM, T^*M)$, where h^t is the composition $TM \overset{i}{\hookrightarrow} T^{**}M \overset{h^*}{\to} T^*M$. On the other side, since $g \in \mathcal{M}$, as a Riemannian metric is a fiberwise inner product on TM it induces a fiberwise inner product on any tensor bundle over M, in particular on $\mathcal{S}^2(T^*M)$. This is, in fact $< \cdot, \cdot >= trace(g^{-1} \cdot g^{-1} \cdot)$. For the metric G_g, instead of the notation above, we will use the classical notations from Riemannian geometry

(the '$\sharp$' symbol demotes the 'sharp' isomorphism induced by the metric g so we will omite to put indices as $\sharp_g$):

$$G_g : T_g\mathcal{M} \times T_g\mathcal{M} \to \mathbb{R},\ G_g(h,k) = \int_M \sum_{i=1}^{n} h(k(E_i)^\sharp, E_i) d\nu_g,$$

Where (E_i) denotes a local field of orthonormal frames.

Theorem 3.2. *The Riemannian metric G_g is defined on the tangent space $T_g\mathcal{M}_{finite} = \Gamma_g$.*

Proof. Since $h \in \Gamma_g$, we have $h \in \Gamma_g^0$. This means that $(\exists)\lambda \in \mathbb{R}_+$ s.t. $\lambda g \leq h \leq \lambda g$. Because of equation (3), we have that $-|h|_g^0 g \leq h \leq |h|_g^0 g$. Hence, as in [PA-cones], in particular,

$$-|h|_g^0 g(k(E_i)^\sharp, E_i) \leq h(k(E_i)^\sharp, E_i) \leq |h|_g^0 g(k(E_i)^\sharp, E_i),\ i = \overline{1,n};$$

By summation, we have

$$-|h|_g^0 \sum_{i=1}^{n} g(k(E_i)^\sharp, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^\sharp, E_i) \leq |h|_g^0 \sum_{i=1}^{n} g(k(E_i)^\sharp, E_i),$$

and this means

$$-|h|_g^0 \sum_{i=1}^{n} k(E_i, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^\sharp, E_i) \leq |h|_g^0 \sum_{i=1}^{n} k(E_i, E_i). \tag{7}$$

But $k \in \Gamma_g$, so we have $k \in \Gamma_g^0$. This means that $(\exists)\lambda \in \mathbb{R}_+$ s.t. $\lambda g \leq k \leq \lambda g$. As above, $-|k|_g^0 g \leq k \leq |k|_g^0 g$, and in particular

$$-|k|_g^0 g(E_i, E_i) \leq k(E_i, E_i) \leq |k|_g^0 g(E_i, E_i),\ i = \overline{1,n};$$

By summation

$$-|k|_g^0 \sum_{i=1}^{n} g(E_i, E_i) \leq \sum_{i=1}^{n} k(E_i, E_i) \leq |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i). \tag{8}$$

From equations (7) and (8) follows that

$$-|h|_g^0 |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i) \leq \sum_{i=1}^{n} h(k(E_i)^\sharp, E_i) \leq |h|_g^0 |k|_g^0 \sum_{i=1}^{n} g(E_i, E_i)$$

and so

$$-n|h|_g^0 |k|_g^0 \leq \sum_{i=1}^{n} h(k(E_i)^\sharp, E_i) \leq n|h|_g^0 |k|_g^0.$$

Now, by integrating with respect to the measure ν_g

$$-n|h|_g^0 |k|_g^0 Vol(M,g) \leq G_g(h,k) \leq n|h|_g^0 |k|_g^0 Vol(M,g)$$

Q.E.D.

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